# %Chapter Fundamental Knowledge

%introduction

## Before the discussion about coupling fibers to waveguides it is necessary to present some basic knowledge involved in this work. In this chapter lens theory and fiber optics will be at first introduced. Then the description of Gaussian beam is helpful for understanding some terms, such as spot size, in the following research.

## In order to analyze the simulations results from CST MWS, we also have to present Finite Integration Technique (FIT), which is implemented in CTS MWS. At the end of this chapter S-Parameters will be described because the $|S\_{21}|$ is generally used to measure the coupling efficiency.

## %\section{Optics}

In this part some basic concepts of geometric optics are introduced.

### \subsectioin{Lens Theory}

### %Lens optics

### Lenses are widely used in optics. In this work lenses are used for coupling fibers to photonic waveguide. No matter what kind of application they all base on primary rules: lens theory.\\

### One important function of lenses is focus. It is a common experience by using a magnifying glass to set small pieces of paper or leaves on fire with sunlight. In this scene the lens of the magnifying glass focuses all incoming sun rays at a single spot. This spot is called focal point and the distance between the focal point and the center of the lens is called focal length $f$. The dimensions of this spot and the focal length are two primary characteristics parameters of the lens and values of them depend on radii of lenses’ surfaces and on the refractive index of the material the lens is made from. \\

### \begin{figure}[!ht]

### \centering

### \includegraphics[width=0.8\textwidth]{bilder/lens\_define}

### \caption{The quantities define of singlet lenses \cite{lens\_theory\_LC\_Ltd}.}

### \label{fig:lens\_define}

### \end{figure}

### In order to understand more specifics about lenses Fig. \ref{fig:lens\_define} presents a singlet lens with geometric parameters. 'The optical axis (O-O') of the lens is a line passing through the centers of curvature of the two spherical lens surfaces.'

### Rays A and B, parallel with the optic axis (O-O'), are casted from different two sides through the lens and respectively cross the axis at their front and back focal points $F1$ and $F2$. The front and back principal points $H1$ and $H2$ are the intersections of the optical axis with the front and back principal surfaces. Points $V1$ and $V2$ are called the front and back vertices respectively\cite{lens\_theory\_LC\_Ltd}. Some important quantities are defined in Tab. \ref{tab:lens\_quantities}. \\

### \begin{table}[!ht]

### \centering

### \caption{Important Quantities for Singlet Lenses Immersed in Air. The distance $t\_{c}$ is the center thickness of the lens $V1V2$, $R\_{1}$ and $R\_{2}$ are respective radii of front and back principal surfaces\cite{lens\_theory\_LC\_Ltd}.}

### \begin{tabular}{|c|c|c|}

### \hline

### \textbf{Symbol}&\textbf{Description}&\textbf{Formula}\\

### \hline

### $f$ & \parbox[c]{6cm}{

### \begin{center}

### effective focal length

### \end{center}

### }& $\frac{1}{f}=(n-1)\left[\frac{1}{R\_{1}}-\frac{1}{R\_{2}} \right]+\frac{t\_{c}(n-1)^2}{nR\_{1}R\_{2}}$ \\

### \hline

### $BFD$ &\parbox[c]{6cm}{

### \begin{center}

### back focal distance

### \end{center}

### }& $BFD=f\left[ 1-\frac{t\_{c}(n-1)}{nR\_{1}}\right]$ \\

### \hline

### $FFD$ &\parbox[c]{6cm}{

### \begin{center}

### front focal distance

### \end{center}

### }& $FFD=f\left[ 1+\frac{t\_{c}(n-1)}{nR\_{1}}\right]$ \\

### \hline

### $H2V2$ & \parbox[c]{6cm}{

### \begin{center}

### back vertex to back principal point distance

### \end{center}

### } & $H\_{2}V\_{2}=f-BFD=-f\frac{t\_{c}(n-1)}{nR\_{1}}$ \\

### \hline

### $V1H1$ & \parbox[c]{6cm}{

### \begin{center}

### front vertex to front principal point distance

### \end{center}

### } & $V\_{1}H\_{1}=f-FFD=-f\frac{t\_{c}(n-1)}{nR\_{2}}$ \\

### \hline

### \end{tabular}

### \label{tab:lens\_quantities}

### \end{table}

### The classical singlet lenses are plano convex, plano concave, equiconvex and equiconcave lenses, which are listed in following Tab. \ref{tab:lenses\_focal\_length} with their focal length relation. The performance of a lens is usually estimated by spot size and focal lengths. The definition of the spot size will be declared later in section \ref{sect:gaussian\_beam}. The focal length is the distance from lens center to minimum spot. But this idea is not exact in any case. Here we can testify this thought by Fig. \ref{fig:focal\_length} and following equations (\ref{eq:snell\_focal}-\ref{eq:focal\_length}).\\

### \begin{table}[!ht]

### \centering

### \caption{Focal length Formulas of Simple Singlet Lenses in Air and

### Radii are considered positive in the formulas below\cite{lens\_theory\_LC\_Ltd}.}

### \begin{tabular}{|c|c|c|}

### \hline

### \textbf{Type}&\textbf{Description}&\textbf{Formula}\\

### \hline

### Plano Convex & \parbox[c]{2.1cm}{\includegraphics[width=2cm]{bilder/plano\_convex}}& $f=\frac{R}{(n-1)}$ \\

### \hline

### Plano Concave &\parbox[c]{2.1cm}{\includegraphics[width=2cm]{bilder/plano\_concave}} & $f=-\frac{R}{(n-1)}$ \\

### \hline

### Equiconvex & \parbox[c]{2.1cm}{\includegraphics[width=2cm]{bilder/equi\_convex}} & $f=\left[\frac{2(n-1)}{R} - \frac{t\_{c}(n-1)^2}{nR^2}\right]^{-1}$ \\

### \hline

### Equiconcave & \parbox[c]{2.1cm}{\includegraphics[width=2cm]{bilder/equi\_concave}} & $f=\left[\frac{2(n-1)}{R} + \frac{t\_{c}(n-1)^2}{nR^2}\right]^{-1}$ \\

### \hline

### \end{tabular}

### \label{tab:lenses\_focal\_length}

### \end{table}

### \begin{figure}[!ht]

### \centering

### \includegraphics[width=0.8\textwidth]{bilder/focal\_length}

### \caption{Schema of refraction of parallel light by plano convex lens.}

### \label{fig:focal\_length}

### \end{figure}

### \begin{equation}

### n\sin(\theta)=\sin(\psi)

### \label{eq:snell\_focal}

### \end{equation}

### \begin{equation}

### \phi=\psi-\theta

### \label{eq:psi\_phi}

### \end{equation}

### \begin{align}

### L&=R\sin(\theta)\cot(\phi) \nonumber\\

### &=R\sin(\theta)\frac{\cos(\psi-\theta)}{ \sin(\psi-\theta)} \nonumber\\

### &= R\sin(\theta)\frac{\cos(\psi)\cos(\theta)+\sin(\psi)\sin(\theta)}{\sin(\psi)\cos(\theta)-\cos(\psi)\sin(\theta)} \nonumber\\

### &= R\sin(\theta)\frac{\cos(\psi)\cos(\theta)+n\sin^{2}(\theta)}{n\sin(\theta)\cos(\theta)-\cos(\psi)\sin(\theta)} \nonumber\\

### &=R\frac{\cos(\psi)\cos(\theta)+n\sin^{2}(\theta)}{n\cos(\theta)-\cos(\psi)} \nonumber\\

### &=R\frac{\cos(\psi)\cos(\theta)-n\cos^{2}(\theta)+n}{n\cos(\theta)-cos(\psi)} \nonumber\\

### &=R \left[-\cos(\theta)+\frac{n}{n\cos(\theta)-\sqrt{1-n^{2}\sin^{2}(\theta)}} \right]

### \label{eq:focal\_length}

### \end{align}

### In Fig. \ref{fig:focal\_length} there is a plano convex lens with radius $R$ and index $n$. The optic axis (O-O') passes through the lens center. The light source $a$ emits a ray parallel with the optic axis (O-O') and at the point $b$ on the lens surface is refracted. At last the refracted ray cross the optic axis (O-O') at point F. $\theta$ is the input angle, $\psi$ the output angle and $\phi$ the angle from output ray to optic axis (O-O'). According the SNELL's LAW we get the relation between $\theta$ and $\psi$ in (\ref{eq:snell\_focal}). $\phi$ and $\psi$ has the relation like (\ref{eq:psi\_phi}). Then the distance $L$ from point H to F is given by (\ref{eq:focal\_length}). When $\theta$ is close to 0 like (\ref{eq:focal\_length\_app}) $L$ is right equal the formula of Plano Convex in Tab. \ref{tab:lenses\_focal\_length}. So this formula of focal length is only valid for a small angle lens.

### \begin{equation}

### L=R\left[ -1+\frac{n}{n-1}\right]=\frac{R}{n-1}

### \label{eq:focal\_length\_app}

### \end{equation}

### \begin{figure}[!ht]

### \centering

### \includegraphics[width=0.9\textwidth]{bilder/min\_max\_spot}

### \caption{Schema to estimating the minimum spot location \cite{lens\_theory\_LC\_Ltd}.}

### \label{fig:min\_max\_spot}

### \end{figure}

### Following we will discuss the focal length of a wide-angle lens. \cite{lens\_theory\_LC\_Ltd} has referred that the minimum spot lies between the maeginal plane and paraxial focal plane. All distances which are in following discussed base on the assumption that the back vertex (V2) of the lens is regarded as origin. In Fig.\ref{fig:min\_max\_spot} there are 'geometrical traces of 25 rays in the focal region 100mm focal length plano convex lens (n=1.515).' 'The rays are launched parallel to the axis (O-O') and equally spaced in a region above and below the axis in a plane containing the axis (\textbf{meridional plane})'. The marginal plane (\textbf{MP}) goes through the focal point of marginal rays. The paraxial focal plane (\textbf{PP}) goes through the focal point of paraxial rays. 'The distance from \textbf{PP} to \textbf{MP} is the \textbf{longitudinal spherical aberration LAm}'. The minimum spot (\textbf{MS}) is located at the plane, which is approximately 3/4 LAm back toward the lens from the \textbf{PP}.\\

### \subsectioin{Optical waveguides and Fibers}

%Optical waveguides

%Optical waveguides

For the transmission of optical signal optical waveguides are applied. The general waveguides are semiconductor waveguide and optical fibers.

Fig. \ref{fig:semi\_waveguides} shows two semiconductor waveguides commonly used in integrated optics. Rib waveguide is composed of a rib guide on a substrate ($n=n\_{2}$). Buried waveguide is a high index guide ($n=n\_{1}$) surrounded by low index cladding ($n=n\_{2}$).\\

\begin{figure}

\centering

\subfigure[Rib waveguide]{

\includegraphics[width=0.4\textwidth]{bilder/approxmate\_waveguide}

\label{fig:semi\_rib\_waveguide}

}

\hfill

\subfigure[Buried waveguide]{

\includegraphics[width=0.4\textwidth]{bilder/buried\_waveguide}

\label{fig:semi\_buried\_waveguide}

}

\caption{Schema of semiconductor waveguides}

\label{fig:semi\_waveguides}

\end{figure}

Optical fibers are widely used for telecommunication and data networks. The Fig.\ref{fig:opticfiber} presents a simplest optical fiber and how lights propagate in the fiber . Optical fiber typically consists of a transparent core with index $n\_{1}$ surrounded by a transparent cladding material with a lower index of refraction $n\_{2}$.

\begin{figure}[httbp]

\centering

\includegraphics[width=0.8\textwidth]{bilder/opticfiber}

\caption{linght refraction in optic fibers}

\label{fig:opticfiber}

\end{figure}

\textbf{ Total Reflection}\\

\begin{figure}[!ht]

\centering

\subfigure[totalreflection]{

\includegraphics[width=0.3\textwidth]{bilder/totalreflection01}

\label{fig:totalreflection01}

}

\hfill

\subfigure[Buried waveguide]{

\includegraphics[width=0.3\textwidth]{bilder/totalreflection02}

\label{fig:totalreflection02}

}

\hfill

\subfigure[Buried waveguide]{

\includegraphics[width=0.3\textwidth]{bilder/totalreflection03}

\label{fig:totalreflection03}

}

\caption{Total reflection}

\label{fig:totalreflection}

\end{figure}

Whatever semiconductor waveguides or optical fibers, the principle of the light propagation in waveguides is total reflection. The principle of the total reflection is explained in \cite{optical\_waveguides\_fibers} with Snell's law. In Fig.\ref{fig:totalreflection01} the input light strikes the boundary between two different isotropic media with respective refractive index $n\_{1}$ and $n\_{2}$. Where $theta\_{1}$ is incidence angle, $\theta\_{2}$ refractive angle and $\theta\_{r}$ reflective angle. Through SNELL's Law there are relations (\ref{eq:snell}-\ref{eq:reflection}). For $n\_{1}<n\_{2}$ there is always a relation $\theta\_{1}>\theta\_{2}$. If the refractive indexes has the relation $n\_{1}>n\_{2}$, then the incidence angle $\theta\_{1}$ is narrower than the refractive angle $\theta\_{2}$ like Fig. \ref{fig:totalreflection02}. If the incidence angle is increased wider than a critical angle $\theta\_{c}$ (\ref{eq:critical\_angle}) there will be no light passing through the boundary and all of the lights are reflected like Fig. \ref{fig:totalreflection03}. This phenomenon is so called total reflection.

\begin{align}

n\_{1}sin\theta\_{1}&=n\_{2}sin\theta\_{2}

\label{eq:snell}\\

\theta\_{1}=\theta{r}

\label{eq:reflection}

\end{align}

\begin{equation}

\theta\_{c}=arcsin(\frac{n\_{2}}{n\_{1}})

\label{eq:critical\_angle}

\end{equation}

%%dispersion

\textbf{ Numerical Aperture }\\

%Numerical Aperture

Another important character of optical waveguides is numerical aperture. Back to the Fig.\ref{fig:opticfiber} the incidence beam originate from the air into the fiber. There is a maximum coupling angle, so that the beam can be guided under the total reflecting conditions. Its sinus value (\ref{eq:NA}) is called \textbf{Numerical Aperture (NA)}, which indicate the acceptable range of ray beams.

\begin{align}

sin\theta\_{i}&=\frac{n\_{1}}{n\_{0}}sin(90^{o}-\theta\_{c})=n\_{1}cos\theta\_{c} \nonumber\\

&=n\_{1}\sqrt{1-sin^{2}\theta\_{c}}=n\_{1}\sqrt{1-\left(\frac{n\_{2}}{n\_{1}}\right)^2}=\sqrt{n^2\_{1}-n^2\_{2}}

\label{eq:NA}

\end{align}

\textbf{Mode of the waveguide}\\

‘An eigenmode $m$ of a waveguide structure is a propagation or evanescent wave which maintains its transverse shape during propagation '\cite{integrated\_optics}. The eigenmode of a waveguide can be presented as (\ref{eq:e\_eigenmode}-\ref{eq:h\_eigenmode}).

\begin{align}

E^{m}(r\_{t},z)&=E^{m}\_{0}(r\_{t})e^{jk\_{m}z}

\label{eq:e\_eigenmode}\\

H^{m}(r\_{t},z)&=H^{m}\_{0}(r\_{t})e^{jk\_{m}z}

\label{eq:h\_eigenmode}

\end{align}

Where $k\_{m}$ is the propagation constant. When the request $k\_{0}n\_{1}>k\_{m}>k\_{0}n\_{2}$ is matched this mode is a guided mode for the waveguides \cite{script\_FT\_TET}. In this work the coupling bases on the fundamental mode.

## \Section{Gaussian Beam}

## In the world there is no natural source of paraxial rays. Each beam of light can be considered originating from a simple origin: point light source, which emits anisotropic light in all directions. Thus a normal light source cannot provide perfect focused beams for optical applications. TEM$\_{00}$ mode of a laser source is a perfect plane wave with Gaussian transverse irradiance profile\cite{CVI\_Melles\_Griot\_Technical\_Guide}. Therefore the laser light is considered as a beam propagating in a well-defined direction with limited spreading in transversal dimensions.\\

## In \cite{ script\_FT\_TET} the transversal e-field components of Gaussian beam is assumed as (\ref{eq:gaussian\_01}).

## \begin{equation}

## E\_{x}=\psi(x,y,z)e^{-jk\_{0}nz}

## \label{eq:gaussian\_01}

## \end{equation}.

## We insert (\ref{eq:gaussian\_01}) into wave equation (\ref{ eq:wave\_equation})

## \begin{equation}

## \Delta E\_{x}+k^{2}\_{0}n^{2}E\_{x}=0

## \label{eq:wave\_equation}

## \end{equation}.

## and obtain partial differential form (\ref{eq:gaussian\_02}). Because the paraxial beams spread slowly in transversal direction due to the z-axis, the term $\frac{\partial ^{2}\psi}{\partial z^2}$ is very smaller than other terms. Thus equation (\ref{eq:gaussian\_02}) become an approximation (\ref{eq:gaussian\_03}).

## \begin{equation}

## \frac{\partial ^{2}\psi}{\partial x^2}+\frac{\partial ^{2}\psi}{\partial y^2}+\frac{ \partial ^{2}\psi}{\partial z^2}-2jk\_{0}n\frac{\partial\psi}{\partial z}=0

## \label{eq:gaussian\_02}

## \end{equation}

## \begin{equation}

## \frac{ \partial ^{2}\psi}{\partial x^2}+\frac{\partial ^{2}\psi}{\partial y^2}-2jk\_{0}n\frac{\partial\psi}{\partial z}=0

## \label{eq:gaussian\_03}

## \end{equation}

## The equation(\ref{eq:gaussian\_03}) can also be transformed into cylinder coordinate and become (\ref{eq:gaussian\_04})

## \begin{equation}

## \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)+\frac{1}{r^2}\frac{ \partial ^{2}\psi}{\partial \phi^2}-2jk\_{0}n\frac{\partial\psi}{\partial z}=0

## \label{eq:gaussian\_04}

## \end{equation}

## (\ref{eq:gaussian\_05}) is one general solution of (\ref{eq:gaussian\_04})

## \begin{equation}

## \psi(r,z)=\psi\_{0}exp\left(-j\left[P(z)+2 \frac{ k\_{0}n}{2q(z)}r^2\right]\right)

## \label{eq:gaussian\_05}

## \end{equation}

## Where $P(z)$ and $q(z)$ meet the equation (\ref{eq:gaussian\_04}) for any $r$. The term $q(z)$ has also the relation(\ref{eq:gaussian\_06}) with some physical variables $w(z)$ and $R(z)$

## \begin{equation}

## \frac{1}{q(z)}=\frac{1}{R(z)}-\frac{j\lambda}{\pi nw^{2}(z)}

## \label{eq:gaussian\_06}

## \end{equation}

## Where $w(z)$ is the $1/e^2$ irradiance radius due to propagating distance z and $R(z)$ is the wavefront radius of curvature due to propagating distance z

## \begin{equation}

## w(z)=w\_{0}\sqrt{1+(\frac{\lambda z}{n\pi w^{2}\_{0}})^{2}}

## \label{eq:gaussian\_07}

## \end{equation}

## \begin{equation}

## R(z)=z\left[1+(\frac{\lambda z}{n\pi w^{2}\_{0}})^{2}\right]

## \label{eq:gaussian\_08}

## \end{equation}

## If (\ref{eq:gaussian\_06}) is inserted in (\ref{eq:gaussian\_05}) then we can get a Gauss form result like (\ref{eq:gaussian\_09}). That means the field of a Gaussian beam is in transversal dimensions a Gaussian distribution like Fig. \ref{fig:gaussian\_verteilung}.

## \begin{equation}

## \psi(r,z)=w\_{0}exp\left(-\frac{r^2}{w^2\_{0}}\right)

## \label{eq:gaussian\_09}

## \end{equation}

## \begin{figure}[!ht]

## \centering

## \includegraphics[width=0.6\textwidth]{bilder/gussian\_verteilung}

## \caption{ransversal profile of the Gaussian beam is the curve of Gaussian functions. At the beam center the field has the maximum value. At the distance of $w(z)$ form the center the field decline to $1/e$ of the peak value.}

## \label{fig:gaussian\_verteilung}

## \end{figure}

## According to definitions of $w(z)$ and $R(z)$ in (\ref{eq:gaussian\_07}-\ref{eq:gaussian\_08}) the longitudinal profile of Gaussian beams can be drawn as Fig. \ref{fig:gussian\_profile}.

## \begin{figure}[!ht]

## \centering

## \includegraphics[width=0.9\textwidth]{bilder/gussian\_profile}

## \caption{The lateral profile of Gaussian beams presents the domain of beam power from peak value to $1/e^2$ peak value. $w(z)$ represents the radius of this profile. $R(z)$ is the radius of wavefronts.}

## \label{fig:gussian\_profile}

## \end{figure}

## From the beam axis to the distance of beam radius $w(z)$ the intensity drops to $1/e2 (\sim13.5\%)$ of the maximum value. That means, a hard aperture with radius $w$ can transmit $\sim86.5\%$ of the optical power.

## %Spot Size

## \textbf{Spot Size}\\

## Another important characteristic of Gaussian beams is \textbf{Spot Size}. In a cross-section of a Gaussian beam the beam intensity is approximately distributed as Gaussian function. The spot size is the diameter of the domain at whose edge the value of the electrical field intensity decays to $1/e$ of the peak value, otherwise the energy density decays to $1/e^2$ of the peak value.

## surrounding

## \section{FIT}

%FIT

%FIT

The Finite Integration Theory(\textbf{FIT}), which was introduced at 1976 by Thomas Weiland\cite{FIT\_discrete\_method} to solving the electromagnetical problems, is a numerical simulation method to discretize the integral form of fundamental Maxwell's functions.\\

The first thought of the \textbf{FIT} is to discretize the calculating volume. There is lot of methods to accomplish this mission. One common way is as Fig.\ref{fig:discretization\_material} to discetize a brick-shape object into a \textbf{Cartesian Grid} (primary cell \textbf{G}). Of cause it is possible to construct grid volume in cylinder coordinate or any other types of coordinates\cite{FIT\_triangular\_discretization,FDTD\_nonorthogonal\_grids} for simplifying the calculation depending on the real dimensions of the object. In general \cite{script\_FeldSim} indicate the computational grid as (\ref{eq:grid}).\\

\begin{align}

G=

(u(i),v(j),w(k))\in \mathbb{R}^3|&u(1)\leq u(i)\leq u(I),\nonumber\\

&v(1)\leq v(j)\leq v(J),\nonumber\\

&w(1)\leq w(k)\leq w(K)

\label{eq:grid}

\end{align}

$u(i)$,$v(j)$,$w(k)$ can be three components of any 3D coordinate systems. In following discussion all computations base on \textbf{Cartesian Coordinate system}.\\

%fig: discretization of the material

\begin{figure}[!ht]

\centering

\includegraphics[width=0.8\textwidth]{bilder/grid\_volum}

\caption{A Cartesian descretized Grid Volum\cite{script\_FeldSim}.}

\label{fig:discretization\_material}

\end{figure}

For calculations following primary elements are defined in \cite{script\_FeldSim}:

\begin{itemize}

\item Points $P(i,j,k)$

\item Elemental Edge:

\begin{align}

\Delta u(i)&=\overline{u(i)u(i+1)} \quad with \quad 1\leq i \leq I-1, \nonumber\\

\Delta v(j)&=\overline{v(j)v(j+1)} \quad with \quad 1\leq j \leq J-1, \nonumber\\

\Delta w(k)&=\overline{w(k)w(k+1)} \quad with \quad 1\leq k \leq K-1

\label{eq:discrete\_edge}

\end{align}

\item Elemental Planes

\begin{align}

A\_{u}(j,k)&=\Delta v(j)\Delta w(k) \quad with \quad 1\leq i \leq I-1,1\leq j \leq J-1\nonumber\\

&A\_{v}(i,k),A\_{w}(i,j) \quad analogous

\label{eq:discrete\_plane}

\end{align}

\item Elemental Cells

\begin{align}

V(i,j,k)=\Delta u(i)\Delta v(j)\Delta w(k) \quad with &\quad 1\leq i\leq I-1,\nonumber\\

&1\leq j\leq J-1,\nonumber\\

&1\leq k\leq K-1

\label{eq:discrete\_cell}

\end{align}

\end{itemize}

A new index system has been introduced for numerical convenience:

\begin{equation}

n=1+(i-1)\cdot M\_{u}+(j-1)\cdot M\_{v}+(k-1)\cdot M{w}

\label{eq:discrete\_index}

\end{equation}

Where $M\_{u}=1,M\_{v}=I,M\_{w}=I\cdot J$. We can reindex primary elements with new index rules $n$ as following:

\begin{itemize}

\item Points $P(n) \quad with \quad 1\leq n \leq N\_{p}$

\item Elemental Edge:

%$\delta u(i)=\overline{u(i)u(i+1)} with 1\leq{i}\leq{I-1},$

\begin{align}

\Delta u(n)&=\overline{u(n)u(n+1)} \quad with \quad 1\leq n \leq N\_{p}, \nonumber\\

\Delta v(n)&=\overline{v(n)v(n+1)} \quad with \quad 1\leq n \leq N\_{p}, \nonumber\\

\Delta w(n)&=\overline{w(n)w(n+1)} \quad with \quad 1\leq n \leq N\_{p}

\label{eq:discrete\_edge\_n}

\end{align}

\item Elemental Planes

\begin{align}

A\_{u}(n)&=\Delta v(n)\Delta w(n) \quad with \quad 1\leq n\leq N\_{p}\nonumber\\

&A\_{v}(n),A\_{w}(n) \quad analogous

\label{eq:discrete\_plane\_n}

\end{align}

\item Elemental Cells

\begin{equation}

V(n)=\Delta u(n)\Delta v(n)\Delta w(n) \quad with \quad 1\leq n\leq N\_{p}

\label{eq:discrete\_cell\_n}

\end{equation}

\end{itemize}

Where $N\_{p}$ is the total number of the grid points:

\begin{equation}

N\_{p}=I\cdot J\cdot K

\label{eq:np}

\end{equation}

The next step is to discretize Maxwell's equations in their integral form(\ref{eq:maxwell\_1}-\ref{eq:maxwell\_4})

\begin{align}

\oint\_{\partial A}\vec{E}\cdot\mathrm{d}\vec{s}&=

-\frac{\mathrm{d}}{\mathrm{d}t}\int\_{A}\vec{B}\cdot\mathrm{d}\vec{A}

\label{eq:maxwell\_1}\\

\oint\_{\partial A}\vec{H}\cdot\mathrm{d}\vec{s}&=

\int\_{A}(\frac{\partial\vec{D}}{\partial t}+\vec{J})\cdot\mathrm{d}\vec{A}

\label{eq:maxwell\_2}\\

\oint\_{\partial V}\vec{D}\cdot\mathrm{d}\vec{A}&=

\int\_{V}\rho\mathrm{d}V

\label{eq:maxwell\_3}\\

\oint\_{\partial V}\vec{B}\cdot\mathrm{d}\vec{A}&=0

\label{eq:maxwell\_4}

\end{align}

and additional equations (\ref{eq:maxwell\_5}-\ref{eq:maxwell\_7}).:

\begin{align}

\vec{D}&=\epsilon\_{0}\epsilon\_{r}\vec{E}

\label{eq:maxwell\_5}\\

\vec{B}&=\mu\_{0}\mu\_{r}\vec{H}

\label{eq:maxwell\_6}\\

\vec{J}&=\kappa\vec{E}+\vec{J\_{s}}

\label{eq:maxwell\_7}

\end{align}

%%%

%%law inductive

Following is a demonstration of the discretized form from the inductive law (\ref{eq:maxwell\_1}). Considering the path integral in a single element edge like Fig. \ref{fig:FIT\_max\_integral1},the left side of (\ref{eq:maxwell\_1}) can be presented as (\ref{eq:inductive\_left}).

\begin{figure}[!ht]

\centering

\includegraphics[width=0.45\textwidth]{bilder/FIT\_max\_integral1}

\caption{ Path integral along edges of one single elemental plane $A\_{u}(n)$\cite{ script\_FeldSim}}

\label{fig:FIT\_max\_integral1}

\end{figure}

\begin{figure}[!ht]

\centering

\subfigure[Discrete electric field strength components distribute at edges of grids.]{

\includegraphics[width=0.4\textwidth]{bilder/FIT\_max\_integral2}

\label{fig:FIT\_max\_integral2}

}

\hfill

\subfigure[Discrete magnetic flux density components distribute at planes of grids]{

\includegraphics[width=0.4\textwidth]{bilder/FIT\_max\_integral3}

\label{fig:FIT\_max\_integral3}

}

\caption{Allocations of components at grids\cite{ script\_FeldSim}.}

\end{figure}

\begin{equation}

\int\_{C}\vec{E}\cdot d\vec{s}

=\underbrace{\int\_{\Delta v(n)}\vec{E}\cdot d\vec{s}}\_{\widehat{e}\_{v}(n)}

+\underbrace{\int\_{\Delta w(n+M\_{v})}\vec{E}\cdot d\vec{s}}\_{\widehat{e}\_{w}(n+M\_{v})}

-\underbrace{\int\_{\Delta v(n+M\_{w})}\vec{E}\cdot d\vec{s}}\_{\widehat{e}\_{v}(n+M\_{w})}

-\underbrace{\int\_{\Delta w(n)}\vec{E}\cdot d\vec{s}}\_{\widehat{e}\_{w}(n)}

\label{eq:inductive\_left}

\end{equation}

Where $\widehat{e(n)}$ is so called electric grid voltage and has the following relation with electric field strength $e(n)$

\begin{equation}

e\_{v}(n)=\frac{\widehat{e}\_{v}(n)}{\Delta v(n)}

\label{eq:e\_field}

\end{equation}

Meanwhile the right side of (\ref{eq:maxwell\_1}) approximates to (\ref{eq:inductive\_right})

\begin{equation}

-\iint\_{A\_{u}(n)}\frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{A}

=-\iint\_{A\_{u}(n)}\frac{\partial B^{\*}\_{u}}{\partial t}\cdot\mathrm{d}A

\approx -\frac{\partial}{\partial t}b\_{u}(n)A\_{u}(n)

\label{eq:inductive\_right}

\end{equation}

Where $A\_{u}(n)=\Delta v(n)\Delta w(n)$ and $b\_{u}(n)$ (Fig.\ref{fig:FIT\_max\_integral2}) is magnetic flux density

\begin{equation}

b\_{u}(n)=\frac{\widehat{\widehat{b}}\_{u}(n)}{\Delta A\_{u}(n)}

\label{eq:b\_flux\_density}

\end{equation}

and magnetic grid flux $\widehat{\widehat{b}}\_{u}(n)$ (\ref{eq:mag\_fluxe})

\begin{equation}

\widehat{\widehat{b}}\_{u}(n)=\iint\_{A\_{u}(n)}\vec{B}\cdot\mathrm{d}\vec{A}

\label{eq:mag\_fluxe}

\end{equation}

From equations (\ref{eq:inductive\_left}-\ref{eq:mag\_fluxe}) we obtain the difference form (\ref{eq:inductive\_integral}) of the inductive equation at one single elemental plane:

\begin{equation}

\widehat{e}\_{v}(n)+\widehat{e}\_{w}(n+M\_{v})-\widehat{e}\_{v}(n+M\_{w})-\widehat{e}\_{w}(n)=-\frac{\partial}{\partial t}\widehat{\widehat{b}}\_{u}(n)

\label{eq:inductive\_integral}

\end{equation}

i.e.,

\begin{multline}

\Delta v(n)e\_{v}(n)+\Delta w(n+M\_{v})e\_{w}(n+M\_{v})\\

-\Delta v(n+M\_{w})e\_{v}(n+M\_{w})-\Delta w(n)e\_{w}(n)=-A\_{u}(n)\frac{\partial}{\partial{t}}b\_{u}(n) \hspace{5cm}

\label{eq:inductive\_sample}

\end{multline}

By merging electric field-strength $e(n)$ and magnetic flux density $b(n)$ of all grids into vectors we obtain:

\begin{equation}

e\_{u}:=

\begin{pmatrix}

e\_{u}(1)&\\

\vdots&\\

e\_{u}(N\_{p})&

\end{pmatrix},

e\_{v}:=

\begin{pmatrix}

e\_{v}(1)&\\

\vdots&\\

e\_{v}(N\_{p})&

\end{pmatrix},

e\_{w}:=

\begin{pmatrix}

e\_{w}(1)&\\

\vdots&\\

e\_{w}(N\_{p})&

\end{pmatrix},

e\_{u}:=

\begin{pmatrix}

e\_{u}&\\

e\_{v}&\\

e\_{w}&

\end{pmatrix}.

\label{eq:vector\_e\_field}

\end{equation}

\begin{equation}

b\_{u}:=

\begin{pmatrix}

b\_{u}(1)&\\

\vdots&\\

b\_{u}(N\_{p})&

\end{pmatrix},

b\_{v}:=

\begin{pmatrix}

b\_{v}(1)&\\

\vdots&\\

b\_{v}(N\_{p})&

\end{pmatrix},

b\_{w}:=

\begin{pmatrix}

b\_{w}(1)&\\

\vdots&\\

b\_{w}(N\_{p})&

\end{pmatrix},

b\_{u}:=

\begin{pmatrix}

b\_{u}&\\

b\_{v}&\\

b\_{w}&

\end{pmatrix}.

\label{eq:vector\_m\_flux\_density}

\end{equation}

Expanding the relation (\ref{eq:inductive\_sample}) to all grid cells \cite{FIT\_discrete\_method}\cite{FIT\_discrete\_electrommagnetism}\cite{FIT\_triangular\_discretization}\cite{script\_FeldSim} we arrive at the discrete form of inductive equation:

\begin{equation}

CD\_{s}e=-D\_{A}\frac{\partial}{\partial{t}}b

\label{eq:inductive\_sample\_all}

\end{equation}

Where $D\_{s}$ (\ref{eq:Ds\_matrix}) is elemental edge matrix,

\begin{equation}

D\_{s}=

\begin{pmatrix}

\Delta u(1)&&&&&&&&\\

&\ddots &&&&&&&\\

&&\Delta u(N\_{p})&&&&&&\\

&&&\Delta v(1)&&&&&\\

&&&&\ddots &&&&\\

&&&&&\Delta v(N\_{p})&&&\\

&&&&&&\Delta w(1)&&\\

&&&&&&&\ddots &\\

&&&&&&&&\Delta w(N\_{p})

\end{pmatrix}

\label{eq:Ds\_matrix}

\end{equation}

$D\_{A}$ (\ref{eq:Da\_matrix}) is elemental plane matrix,

\begin{equation}

D\_{A}=

\begin{pmatrix}

\Delta A\_{u}(1)&&&&&&&&\\

&\ddots &&&&&&&\\

&&\Delta A\_{u}(N\_{p})&&&&&&\\

&&&\Delta A\_{v}(1)&&&&&\\

&&&&\ddots &&&&\\

&&&&&\Delta A\_{v}(N\_{p})&&&\\

&&&&&&\Delta A\_{w}(1)&&\\

&&&&&&&\ddots &\\

&&&&&&&&\Delta A\_{w}(N\_{p})

\end{pmatrix}

\label{eq:Da\_matrix}

\end{equation}

$C$ (\ref{eq:C\_matrix}) is \textbf{curl} operator.

\begin{align}

C&=

\begin{pmatrix}

&&\quad\vline &\ddots &-1&\quad\vline &\ddots &+1&\\

&0&\quad\vline & &+1 &\ddots\vline &&-1&\ddots\\

&&\quad\vline &&&\ddots\vline &&&\ddots\\

\hline

\ddots &+1&\quad\vline &&&\quad\vline &\ddots &-1&\\

&-1 &\ddots\vline &&0&\quad\vline &&+1&\ddots\\

&&\ddots\vline &&&\quad\vline &&&\ddots\\

\hline

\ddots &-1&\quad\vline &\ddots &+1&\quad\vline &&&\\

&+1 &\ddots\vline &&-1&\ddots\vline &&0&\\

&&\ddots\vline &&&\ddots\vline &&&

\end{pmatrix}\nonumber\\

&=

\begin{pmatrix}

0&-P\_{w}&P\_{v}\\

P\_{w}&0&-P\_{u}\\

-P\_{v}&P\_{u}&0

\end{pmatrix}

\label{eq:C\_matrix}

\end{align}

Submatrixes $P\_{u},P\_{v},P\_{w}$ are composed of $1,-1,0$ like Fig.\ref{fig:Matrix Px}.\\

\begin{figure}[!ht]

\centering

\includegraphics[width=0.5\textwidth]{bilder/P\_matrix}

\caption{Matrix $P\_{x},(x=u,v,w)$}

\label{fig:Matrix Px}

\end{figure}

Alternative form of the equation(\ref{eq:inductive\_sample\_all})is given:

\begin{equation}

C\widehat{e}=-\frac{\partial}{\partial{t}}\widehat{\widehat{b}}

\label{eq:inductive\_integral\_all}

\end{equation}

%\widehat{e}

Where electric voltage $\widehat{e}$ and magnetic flux $ $\widehat{ \widehat{b}}$ are defined as following:

\begin{equation}

\widehat{e}\_{u}:=

\begin{pmatrix}

\widehat{e}\_{u}(1)&\\

\vdots&\\

\widehat{e}\_{u}(N\_{p})&

\end{pmatrix},

\widehat{e}\_{v}:=

\begin{pmatrix}

\widehat{e}\_{v}(1)&\\

\vdots&\\

\widehat{e}\_{v}(N\_{p})&

\end{pmatrix},

\widehat{e}\_{w}:=

\begin{pmatrix}

\widehat{e}\_{w}(1)&\\

\vdots&\\

\widehat{e}\_{w}(N\_{p})&

\end{pmatrix},

\widehat{e}\_{u}:=

\begin{pmatrix}

\widehat{e}\_{u}&\\

\widehat{e}\_{v}&\\

\widehat{e}\_{w}&

\end{pmatrix}.

\label{eq:vector\_e\_voltage}

\end{equation}

%\widehat{\widehat{b}}

\begin{equation}

\widehat{\widehat{b}}\_{u}:=

\begin{pmatrix}

\widehat{\widehat{b}}\_{u}(1)&\\

\vdots&\\

\widehat{\widehat{b}}\_{u}(N\_{p})&

\end{pmatrix},

\widehat{\widehat{b}}\_{v}:=

\begin{pmatrix}

\widehat{\widehat{b}}\_{v}(1)&\\

\vdots&\\

\widehat{\widehat{b}}\_{v}(N\_{p})&

\end{pmatrix},

\widehat{\widehat{b}}\_{w}:=

\begin{pmatrix}

\widehat{\widehat{b}}\_{w}(1)&\\

\vdots&\\

\widehat{\widehat{b}}\_{w}(N\_{p})&

\end{pmatrix},

\widehat{\widehat{b}}\_{u}:=

\begin{pmatrix}

\widehat{\widehat{b}}\_{u}&\\

\widehat{\widehat{b}}\_{v}&\\

\widehat{\widehat{b}}\_{w}&

\end{pmatrix}.

\label{eq:vector\_m\_flux}

\end{equation}

%divergence equation

\begin{figure}

\centering

\includegraphics[width=0.5\textwidth]{bilder/divergence\_in\_grid}

\caption{This figure discribe the allocation of six magnetic facet fluxes in $G$}

\label{fig:divergence\_G}

\end{figure}

Analogy the divergence equation(\ref{eq:maxwell\_4}) can also be discretized in grid $G$(Fig.\ref{fig:divergence\_G}) and can be described in its difference form:

\begin{equation}

SD\_{A}b=0

\label{eq:divergence\_sample}

\end{equation}

or

\begin{equation}

S\widehat{\widehat{b}}=0

\label{eq:divergence\_integral}

\end{equation}

$S\in \mathbb{R}^{N\_{p}\times 3N\_{p}}$ represent the discrete divergence matrix, which depends on the grid topology just as the discrete $curl-Matrix$ $C$.

%S

\begin{equation}

S=(P\_{u}|P\_{v}|P\_{w})

\label{eq:S\_matrix}

\end{equation}

%Amp\'ere's law

Now considering the discretization of Amp\'ere's law (\ref{eq:maxwell\_2}) its magnetic field strengths pass through the elemental plane along the same path with the magnetic flux density (Fig. \ref{fig:divergence\_G}). Therefore, a new grid (\textbf{Dual Grid}) like Fig. \ref{fig:dual\_grid} is required for this path integral. The second cell $\tilde{G}$ is dual to the primary cell $G$. As primary elements in primary Grid the dual elements are also properly defined in \cite{script\_FeldSim}.

\begin{figure}[!ht]

\centering

\includegraphics[width=0.5\textwidth]{bilder/dual\_grid}

\caption{The allocation of the Primary Grid $G$ and Dual Grid $\tilde{G}$\cite{FIT\_discrete\_electrommagnetism}.}

\label{fig:dual\_grid}

\end{figure}

Like the discretization of Inductive Law Figure. \ref{fig:FIT\_max\_integral4} and Figure. \ref{fig:FIT\_max\_integral5} describe the path integral and surface integral of the equation(\ref{eq:maxwell\_2}) in a \textbf{Dual Grid} and lead to the difference equations (\ref{eq:ampere\_left\_sample}-\ref{eq:ampere\_right}).

\begin{figure}

%\centering

\subfigure[Path Integral of the magnetic flux in dual grid.]{

\includegraphics[width=0.5\textwidth]{bilder/FIT\_max\_integral4}

\label{fig:FIT\_max\_integral4}

}

\hfill

\subfigure[Surface Integral of the electric flux in dual grid.]{

\includegraphics[width=0.4\textwidth]{bilder/FIT\_max\_integral5}

\label{fig:FIT\_max\_integral5}

}

\caption{Discretization of Amp\'ere's law}

\end{figure}

\begin{equation}

\int\_{\tilde{C}}\vec{H}\cdot d\vec{s}=\int\_{\tilde{C}}\mu^{-1}\vec{B}\cdot d\vec{s}\approx

\widehat{h}\_{v}(n)

+\widehat{h}\_{w}(n+M\_{v})

-\widehat{h}\_{v}(n+M\_{w})

-\widehat{h}\_{w}(n)

\label{eq:ampere\_left\_sample}

\end{equation}

\begin{align}

\int\int\_{\tilde{A}\_{u}}\vec{D}\cdot\mathrm{d}\vec{A}\approx &+e\_{u}A\_{1}\epsilon\_{u}(n-M\_{v}-M\_{w}) \nonumber\\

&+e\_{u}A\_{2}\epsilon\_{u}(n-M\_{w}) \nonumber\\

&+e\_{u}A\_{3}\epsilon\_{u}(n) \nonumber\\

&+e\_{u}A\_{4}\epsilon\_{u}(n-M\_{v}) \nonumber\\

&=\bar{\epsilon}\_{u}(n)e\_{u}\tilde{A}\_{u}

\label{eq:ampere\_right}

\end{align}

with

\begin{align}

\widehat{h}\_{v}(n)&=h\_{1}\cdot\frac{\Delta v(n-M\_{v})}{2}+ h\_{2}\cdot\frac{\Delta v(n)}{2}\\

h\_{1}&=\frac{b\_{v}(n)}{\mu (n-M\_{v})}\\

h\_{2}&=\frac{b\_{v}(n)}{\mu (n)}

\label{eq:megnetic\_field}

\end{align}

\begin{align}

A\_{1}&=\frac{A\_{u}(n-M\_{v}-M\_{w})}{4}\\

A\_{2}&=\frac{A\_{u}(n-M\_{w})}{4}\\

A\_{3}&=\frac{A\_{u}(n)}{4}\\

A\_{4}&=\frac{A\_{u}(n-M\_{v})}{4}

\end{align}

\begin{align}

\bar{\epsilon}\_{x}(n)&:=\frac{\int\int\epsilon\mathrm{d}A}{\int\int\mathrm{d}A}\nonumber\\

&=\frac{1}{4\tilde{A}\_{x}(n)}(\epsilon\_{x}(n-M\_{y}-M\_{z})A\_{x}(n-M\_{y}-M\_{z})\nonumber\\

&+\epsilon\_{x}(n-M\_{z})A\_{x}(n-M\_{z}) \nonumber\\

&+\epsilon\_{x}(n)A\_{x}(n) \nonumber\\

&+\epsilon\_{x}(n-M\_{y})A\_{x}(n-M\_{y})

\end{align}

$\bar{\epsilon}$ is average dielectric constant.

The average dielectric matrix is defined:

\begin{equation}

D\_{\epsilon}=Diag(\bar{\epsilon}\_{u}(1),\ldots,\bar{\epsilon}\_{u}(N\_{p}),\bar{\epsilon}\_{v}(1),\ldots,\bar{\epsilon}\_{v}(N\_{p}),\bar{\epsilon}\_{w}(1),\ldots,\bar{\epsilon}\_{w}(N\_{p}))

\label{eq:eps\_matrix}

\end{equation}

By expanding equations (\ref{eq:ampere\_left\_sample}-\ref{eq:eps\_matrix}) we obtain discretized form(\ref{eq:ampere}) .

\begin{equation}

\tilde{C}\tilde{D}\_{s}D\_{\mu^{-1}}b=\tilde{D}\_{A}(D\_{\epsilon}\frac{\mathrm{d}}{\mathrm{dt}}e+D\_{\kappa}e+j)

\label{eq:ampere}

\end{equation}

or

\begin{equation}

\tilde{C}\widehat{h}=\frac{\mathrm{d}}{\mathrm{dt}}\widehat{\widehat{d}}+\widehat{\widehat{j}}\_{L}+\widehat{\widehat{j}}\_{S}

\label{eq:ampere\_sample}

\end{equation}

$\tilde{C}$ represents the $curl-operator$ in dual grid.

\begin{equation}

\tilde{C}=

\begin{pmatrix}

0&-\tilde{P}\_{w}&\tilde{P}\_{v}\\

\tilde{P}\_{w}&0&-\tilde{P}\_{u}\\

-\tilde{P}\_{v}&\tilde{P}\_{u}&0

\end{pmatrix}

\label{eq:dual\_C\_matrix}

\end{equation}

With the help of dual grid cells Gauss' law(\ref{eq:maxwell\_3}) in integral form can be discretized\cite{script\_FeldSim} as following:

\begin{equation}

\tilde{S}\widehat{\widehat{d}}=q

\label{eq:gausslaw}

\end{equation}

or

\begin{equation}

\tilde{S}\tilde{D}\_{A}D\_{\epsilon}e=\tilde{D}\_{V}\rho\_{D}

\label{eq:gausslaw\_sample}

\end{equation}

Here $\rho\_{D}$ is the vector of the charge density in grid cells.

\begin{equation}

\tilde{S}=(\tilde{P}\_{u}|\tilde{P}\_{v}|\tilde{P}\_{w})=(-P\_{u}^{T}|-P\_{v}^{T}|\-P\_{w}^{T})

\label{eq:dual\_S\_matrix}

\end{equation}

## \section{S-Parameter}

%S\_parameter

%S\_parameter

Normally an electrical network can be considered as a 'black box', which contains amounts of interconnected basic electrical circuit components such as resistors, capacitors, inductors and transistors etc. On this 'black box' may exist many ports, which present the entries or exits of the network. In order describe the characteristics of this network H-Parameters are used, which describe the relation between voltages and currents. In Fig. \ref{fig:2\_port\_network} is a 2-port network $V\_{1}$ and $V\_{2}$ are total voltages of both ports; $I\_{1}$ and $I\_{2}$ are total currents of both ports respectively. Relations between voltages and currents are like (\ref{eq:voltage\_current}).

\begin{figure}[!ht]

\centering

\includegraphics[width=0.6\textwidth]{bilder/s\_parameters}

\caption{2-Port-Network \cite{aglient\_s\_parameters}}

\label{fig:2\_port\_network}

\end{figure}

\begin{align}

V\_{1}&=h\_{11}I\_{1}+h\_{12}V\_{2}\\

I\_{2}&=h\_{21}I\_{1}+h\_{22}V\_{2}

\label{eq:voltage\_current}

\end{align}

Where $h\_{11},h\_{12},h\_{21}$ and $h\_{22}$ are H-Parameters and defined in (\ref{eq:h\_parameters1}-\ref{eq:h\_parameters2}).

\begin{align}

h\_{11}&=\frac{V\_{1}}{I\_{1}}|\_{V\_{2}=0}\quad h\_{12}=\frac{V\_{1}}{V\_{2}}|\_{I\_{1}=0}

\label{eq:h\_parameters1}\\

h\_{21}&=\frac{I\_{2}}{I\_{1}}|\_{V\_{2}=0}\quad h\_{22}=\frac{I\_{2}}{V\_{2}}|\_{I\_{1}=0}

\label{eq:h\_parameters2}

\end{align}

But H-Parameters cannot always be valid for the description of microwave circuits. Agilent\cite{aglient\_s\_parameters} has listed some problems of H-Parameters in high frequency:

\begin{itemize}

\item It is not easy to measure the total voltage and total current at the ports of the network.

\item Short and open circuits are not always available for a broad frequency band.

\item For high frequency some circuit are not stable in short or open conditions.

\end{itemize}

Scattering parameters or S-parameters are perfect description of microwave circuit\cite{RF194\_s\_parameters}. In that case traveling waves are applied instead of total voltages and currents. $E\_{i1}$ and $E\_{i2}$ represent incidence waves over left and right ports of the network respectively. $E\_{r1}$ and $E\_{r2}$ are reflective waves. The relation between traveling waves, total voltages and currents has relations (\ref{eq:voltage\_wave1}-\ref{eq:voltage\_wave2}).

\begin{align}

V\_{1}&=E\_{i1}+E\_{r1}\quad V\_{2}=E\_{i2}+E\_{r2}

\label{eq:voltage\_wave1}\\

I\_{1}&=\frac{E\_{i1}-E\_{r1}}{Z\_{0}}\quad I\_{2}=\frac{E\_{i2}-E\_{r2}}{Z\_{0}}

\label{eq:voltage\_wave2}

\end{align}

Traveling waves themselves can be expressed in terms of H-Parameters (\ref{eq:er1}\ref{eq:er2}).

\begin{align}

E\_{r1}&=f\_{11}(h)E\_{i1}+f\_{12}(h)E\_{i2}

\label{eq:er1}

\\

E\_{r2}&=f\_{21}(h)E\_{i1}+f\_{22}(h)E\_{i2}

\label{eq:er2}

\end{align}

Here $f11, f12, f21, f22$ are the network parameters, which indicate the relation between traveling voltages waves and total voltages or currents. Divide both sides of the functions(\ref{eq:er1}-\ref{eq:er2}) by $\sqrt{Z\_{0}}$( $Z\_{0}$ system impedance).A new set of variables are defined:

\begin{align}

a1&=\frac{Ei1}{\sqrt{Z\_{0}}}=\frac{V\_{1}+I\_{1}Z\_{0}}{2\sqrt{Z\_{0}}} \quad a2=\frac{Ei2}{\sqrt{Z\_{0}}}=\frac{V\_{2}+I\_{2}Z\_{0}}{2\sqrt{Z\_{0}}} \\

b1&=\frac{Er1}{\sqrt{Z\_{0}}}=\frac{V\_{1}-I\_{1}Z\_{0}}{2\sqrt{Z\_{0}}} \quad b2=\frac{Er2}{\sqrt{Z\_{0}}}=\frac{V\_{2}-I\_{2}Z\_{0}}{2\sqrt{Z\_{0}}}

\end{align}

So relations of the new variables are give by:

\begin{align}

b\_{1}&=S\_{11}a\_{1}+S\_{12}a\_{2}\\

b\_{2}&=S\_{21}a\_{1}+S\_{22}a\_{2}

\end{align}

or in matrics form:

\begin{equation}

\begin{pmatrix}

b\_{1}&\\

b\_{2}&

\end{pmatrix}

=

\begin{pmatrix}

S\_{11}&S\_{12}\\

S\_{21}&S\_{22}

\end{pmatrix}

\begin{pmatrix}

a\_{1}&\\

a\_{2}&

\end{pmatrix}

\label{eq:s\_matrix}

\end{equation}

The S-Parameters are defined as following:

\begin{align}

S\_{11}&=\frac{b\_{1}}{a\_{1}}|\_{a\_{2}=0}\\

S\_{21}&=\frac{b\_{2}}{a\_{1}}|\_{a\_{2}=0}\\

S\_{22}&=\frac{b\_{2}}{a\_{2}}|\_{a\_{1}=0}\\

S\_{12}&=\frac{b\_{1}}{a\_{2}}|\_{a\_{1}=0}

\end{align}

The phical meaning of the S-Parameters are described as following:

\begin{itemize}

\item $S\_{11}$ is the input port reflection coefficient

\item $S\_{12}$ is the reverse gain

\item $S\_{21}$ is the forward gain

\item $S\_{22}$ is the output port reflection coefficient

\end{itemize}

In another hand, $S\_{21}$ is often used to estimate the transmission ability of a network. Therefore $S\_{21}$ equals the coupling efficiency in this work.